Multi-modal Traffic Assignment with Consideration of Transit Vehicles

Shin-Lai Tien

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Outline

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Introduction

- Traffic assignment is to distribute the traffic demand onto road networks
- The multi-modal assignment models consider the choices among different modes based on the costs the modes involve
- The link travel time functions employed in these models depend on the number of trips of modes, not the number of vehicles

- The link performance will affect the equilibrium flows and then determine the frequency and fleet size of transit service
- The changes in service properties will feedback to the choice of mode
- The objectives of this study are to incorporate the transit vehicle consideration into the assignment model and to modify previous formulation and solution algorithm to obtain the equilibrium flows

Background Information

- Notations
  - Flow variables
    - Path flow: $\hat{f}_l^{rs}$
    - Link flow: $\hat{x}_a = \sum_{rs} \sum_{l} \hat{f}_l^{rs} \hat{\delta}_{a,l}^{rs}$
  - Travel time variables
    - Link travel time: $\hat{t}_a$
    - Path travel time: $\hat{c}_l = \sum_{a \in A} \hat{t}_a \delta_{a,l}^{rs}$
  - Demand variables: $\bar{q}_{rs} = q_{rs} + \hat{q}_{rs}$

Background Information

- The Conventional Formulation

$$MIN \quad Z(x,\hat{q}) = \sum_a \int_0^{x_a} t_a(w)dw + \sum_{r,s} \int_0^{\hat{q}_{rs}} \left(\frac{1}{\theta} \ln \frac{w}{\bar{q}_{rs} - w} + \hat{u}_{rs}\right)dw$$

subject to

$$\sum_k f_k^{rs} = \bar{q}_{rs} - \hat{q}_{rs} \quad , \forall r,s \quad (u_{rs})$$

$$f_k^{rs} \geq 0 \quad , \forall k,r,s$$

- The equivalency conditions
  - User Equilibrium for auto trips

$$(c_l^{rs} - u_{rs}) f_l^{rs} = 0 \qquad c_l^{rs} - u_{rs} \geq 0$$

  - Mode Choice

$$u_{rs} = \frac{1}{\theta} \ln \frac{\hat{q}_{rs}}{\bar{q}_{rs} - \hat{q}_{rs}} + \hat{u}_{rs} \quad \rightarrow \quad \frac{\hat{q}_{rs}}{\bar{q}_{rs}} = \frac{e^{-\theta \hat{u}_{rs}}}{e^{-\theta \hat{u}_{rs}} + e^{-\theta u_{rs}}}$$

where u_{rs} is the minimum travel time for auto between origin r and destination s at equilibrium; $\hat{u}_{rs}$ is the minimum travel time for transit between origin r and destination s at equilibrium

Model Formulation

- Transit Characteristics

Fleet size (no. of veh.)	Frequency (veh/hour)	Headway (min/veh)	Waiting time (min/trip)	Transit Utility and Ridership
↑	↑	↓	↓	↑
↓	↓	↑	↑	↓

$$\text{Headway} = \frac{\text{Round - Trip Time}}{\text{Fleet Size}} = \frac{2\hat{u}_{rs}}{N_{rs}}$$

$$\text{Average Waiting Time} = \frac{1}{2}\text{Headway} = \frac{2\hat{u}_{rs}}{2N_{rs}} = \frac{1}{N_{rs}}\hat{u}_{rs}$$

<u>The transit impedance</u>=(travel time)+(waiting time)= $(1+\frac{1}{N_{rs}})\hat{u}_{rs}$

- The fleet size must be constrained by operational restrictions
  - The upper bound is determined by the road network condition and by the availability of transit operators
  - The lower bound

$$\text{Min. Freq.} = \frac{\text{Max Demand from r to s}}{\text{Bus Capacity}} = \frac{\hat{q}_{rs}}{c}$$

$$N_{MIN} = (\text{Round-Trip Time}) \times (\text{Min Allowable Freq.})$$
$$= \frac{2\hat{u}_{rs}\hat{q}_{rs}}{c}$$

- The Modified Formulation

$$MIN \quad Z(x,\hat{q},N) = \sum_a \int_0^{x_a} t_a(w,N_a)dw + \sum_{r,s} \int_0^{\hat{q}_{rs}} \left[\frac{1}{\theta} \ln \frac{w}{\bar{q}_{rs} - w} + \left(1 + \frac{1}{N_{rs}}\right) \cdot \hat{u}_{rs}(x_a, N_a) \right] dw$$

subject to

$$\sum_k f_k^{rs} = \bar{q}_{rs} - \hat{q}_{rs} \quad ,(u_{rs})$$

$$N_{\min} \le N_{rs} \le N_{\max}$$

$$f_k^{rs} \ge 0$$

- The first-order conditions of this problem is not equivalent to the equilibrium conditions
- The cross-effect of modes in the link travel time function will not come out a solution of equilibrium flows
- The Hessian of the objective function is not symmetric and cannot be sure as a positive definite matrix
- Since for this specific problem there is no known mathematical program the solution of which is the equilibrium flow pattern, we could focus on the direct solution algorithm to solve this problem

Solution Algorithm

- The cross-effect must be relaxed
  - The main problem can be relaxed through solving subproblems
  - In each iteration of subproblem, the cross-effect is fixed, thus the Hessian is diagonal
  - The convergence criteria are based on the similarity of the link flows.
  - The equilibrium flow pattern can be obtained if the algorithm converges
  - A streamlined approach is to get the approximate solution for the next iteration

- At each iteration the main problem is separated
  - The first problem P(I)

$$\underset{N}{MIN} \quad N$$

$$\textit{subject to} \quad N_{\min} \le N_{rs} \le N_{\max} \qquad \textit{where} \quad N_{\min} = \frac{2\hat{u}_{rs}^{n-1}\hat{q}_{rs}^{n-1}}{c}$$

  - The second problem P(II)

$$\underset{x,\hat{q}}{MIN} \quad Z(\mathbf{x},\hat{\mathbf{q}}) = \sum_a \int_0^{x_a} t_a(w,\mathbf{N}^n)dw$$

$$+\sum_{r,s}\int_0^{\hat{q}_{rs}}\left[\frac{1}{\theta}\ln\frac{w}{\bar{q}_{rs}-w}+\left(1+\frac{1}{\mathbf{N}^n}\right)\cdot\hat{u}_{rs}(\mathbf{x}^n,\mathbf{N}^n)\right]dw$$

subject to

$$\sum_k f_k^{rs} = \bar{q}_{rs} - \hat{q}_{rs} \qquad ,(u_{rs})$$

The Procedures of Algorithm

- Step 0: Initialization
  - Decide a feasible solution set (**x,q**), solve $N=N_{min}$, and set n=1
- Step 1: Solve P(II)
  - Step 1-1: Direction finding

$$MIN \quad Z^{n}(g,\hat{v}) = \nabla Z(f^{n},\hat{q}^{n})^{T} \cdot (g,\hat{v})$$

$$= \sum_{r,s}\left[\sum_{k} C_{k}^{rs\,n} g_{k}^{rs} + \left(\sum_{l}\frac{1}{\theta}\ln\frac{\hat{q}_{rs}^{n}}{\bar{q}_{rs}-\hat{q}_{rs}^{n}} + \left(1+\frac{1}{N_{l}^{n}}\right)\cdot \hat{u}_{rs}\right)\hat{v}_{rs}\right]$$

subject to

$$\sum_{k} g_{k}^{rs} = \bar{q}_{rs} - \hat{v}_{rs} \qquad \forall r,s$$

- For each O-D pair (r, s), the objective function will be minimized by assigning the entire O-D volume $\bar{q}_{rs}$ to the path with minimum travel time, either through transit or auto networks
- The solution can be found first by choosing mode
  - If $$u_{rs}^{n} < \frac{1}{\theta} \ln \frac{\bar{q}_{rs} - q_{rs}}{q_{rs}} + (1 + \frac{1}{N_{rs}}) \hat{u}_{rs}^{n}$$

    $\bar{q}_{rs}$ is assigned to the auto network and onto the path with minimum travel time
  - It is equivalent to solve a shortest path problem for auto network

- Step 1-2: Line Search
  - Once the direction of improvement has been decided, we are able to search the optimal step size moving toward this direction by solving the following unconstrained program

$$\underset{0\le\alpha\le1}{MIN} \quad z(\alpha)=\sum_a \int_0^{x_a^n+\alpha(y_a^n-x_a^n)} t_a(w,N_a^n)dw$$

$$+\sum_{r,s}\int_0^{\hat{q}_{rs}^n+\alpha(\hat{v}_{rs}^n-\hat{q}_{rs}^n)}\left[\frac{1}{\theta}\ln\frac{w}{\overline{q}_{rs}-w}+\left(1+\frac{1}{\mathbf{N}^n}\right)\cdot\hat{u}_{rs}(\mathbf{x}^n,\mathbf{N}^n)\right]dw$$

- Step 1-3: Update x, q and set n=n+1

$$x_a^{n+1} = x_a^n + \alpha(y_a^n - x_a^n)$$

$$\hat{q}_{rs}^{n+1} = \hat{q}_{rs}^n + \alpha(\hat{v}_{rs}^n - \hat{q}_{rs}^n)$$

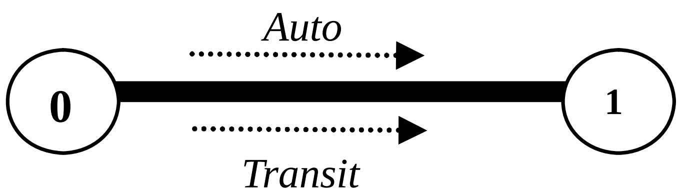
- Step 1-4: Convergence test

$$\max\left\{\left|\frac{x_a^{n+1} - x_a^n}{x_a^n}\right|\right\} \leq \varepsilon_1 \qquad \max\left\{\left|\frac{\hat{q}_{rs}^{n+1} - \hat{q}_{rs}^n}{\hat{q}_{rs}^n}\right|\right\} \leq \varepsilon_2$$

- Step 2: Solve P(I)
  - Update N and go to Step 1
- Step 3: Stop.

Numerical Examples

- Example 1 – Singe OD with Single Link



Total Demand per hour (trips/hr)	$\bar{q}$ =400
Capacity of Transit Vehicle (seats)	c=40
Link travel time function for auto (min)	$t = 15 + 0.05x + 0.5N$
Link travel time function for transit (min)	$\hat{t} = 20 + 0.01x + N$
The coefficient in Logit Choice Model	θ=0.1

- The analytical result

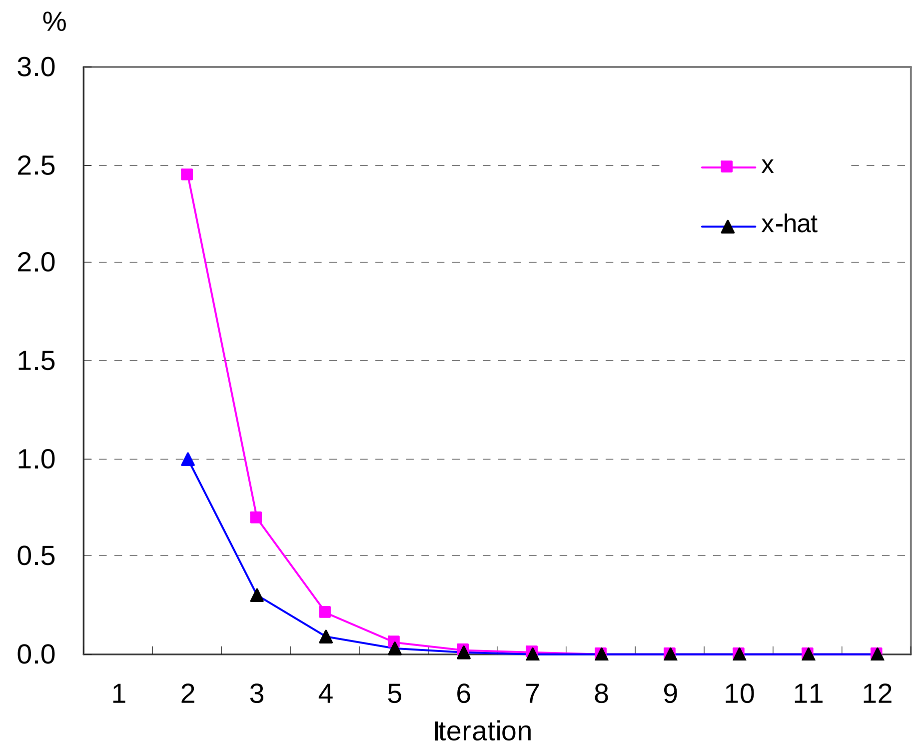
Variable	x	$\hat{x}$	N	t	$\hat{t}$
Optimal value	245.267	154.731	3.323	28.925	25.776

- The Algorithmic Results

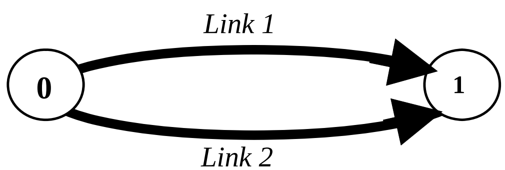
Iteration	Fleet Size	Link Flow		Link Travel Time		Value of Logit Fn.	All-or-Nothing Assignment		Step Size (α)
	N	x	x-hat	t	t-hat		Auto	Transit	
1	4.400000	200.000000	200.000000	27.200000	26.400000	32.400000	400	0	0.1852170
2	3.515195	237.043406	162.956594	28.609768	25.885629	29.501959	400	0	0.0355148
3	3.380261	242.830773	157.169227	28.831669	25.808568	29.093234	400	0	0.0107364
4	3.341079	244.518199	155.481801	28.896449	25.786261	28.976595	400	0	0.0033197
5	3.329108	245.034347	154.965653	28.916271	25.779451	28.941154	400	0	0.0010335
6	3.325395	245.194506	154.805494	28.922423	25.777340	28.930179	400	0	0.0003225
7	3.324238	245.244424	154.755576	28.924340	25.776682	28.926761	400	0	0.0001005
8	3.323877	245.259984	154.740016	28.924938	25.776477	28.925696	400	0	0.0000315
9	3.323764	245.264855	154.735145	28.925125	25.776413	28.925362	400	0	0.0000099
10	3.323729	245.266381	154.733619	28.925183	25.776393	28.925258	400	0	0.0000031
11	3.323718	245.266859	154.733141	28.925202	25.776386	28.925225	400	0	0.0000010
12	3.323714	245.267008	154.732992	28.925208	25.776384	28.925215	400	0	0.0000003
13	3.323713	245.267055	154.732945	28.925209	25.776384	28.925212			

- The Convergence Rate

Iteration	criterion for x	criterion for x-hat
	%	%
1	18.52170295	18.52170295
2	2.441479801	3.551477562
3	0.694897958	1.073636431
4	0.21108775	0.331966803
5	0.065362092	0.10335166
6	0.020358322	0.032245294
7	0.006344633	0.010054474
8	0.00198635	0.003148327
9	0.000622004	0.000985915
10	0.000194786	0.000308753
11	6.10003E-05	9.66913E-05
12	1.91033E-05	3.02806E-05



- Example 2 – Single OD with Parallel Links



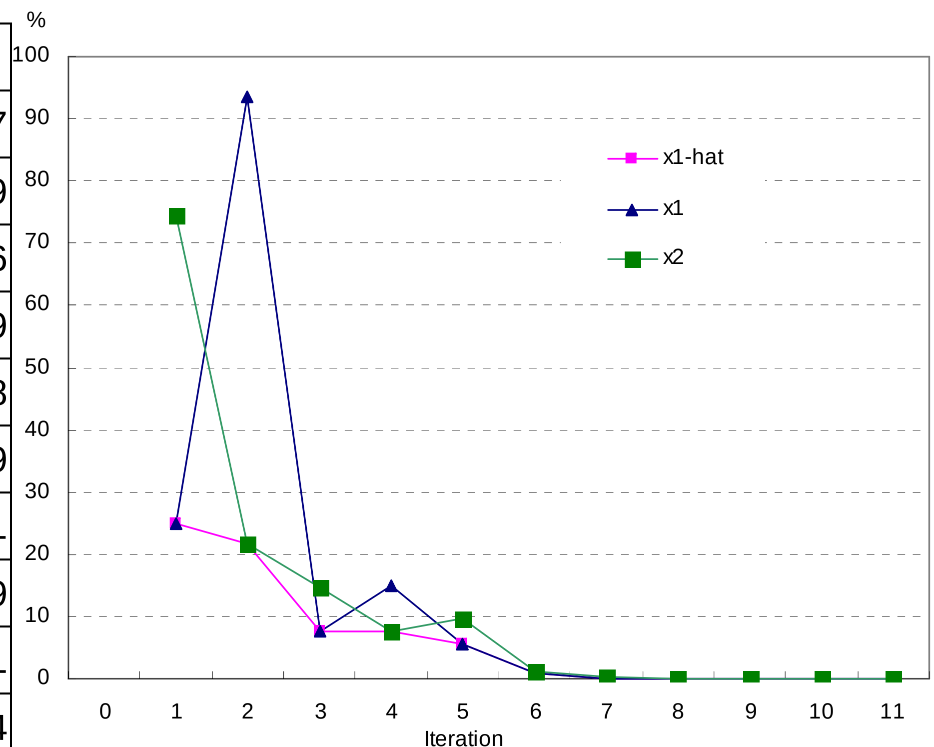
Total Demand per hour (trips/hr)	$\bar{q}_{0,1}=400$
Capacity of Transit Vehicle (seats)	$c=40$
Travel time function for transit on Link 1 (min)	$\hat{t}_1 = 20 + 0.01x_1 + N_1$
Travel time function for auto on Link 1 (min)	$t_1 = 15 + 0.05x_1 + 0.5N_1$
Travel time function for auto on Link 2 (min)	$t_2 = 15 + 0.05x_2$
The coefficient in Logit Choice Model	$\theta=0.1$
The initial values	$\hat{x}_1=200, \quad x_1=100, \quad x_2=100, \quad N=5$

- The Algorithmic Results

Iteration	Variable				Link Travel Time			Mode Choice		Auto Route Choice		alpha
	x1-hat	x1	x2	N	t1-hat	t1	t2	auto	bus	link1	link2	
1	200	100	100	5	26	22.5	20	400	0	0	400	0.247933
2	150.41344	75.20672	174.37984	2.97392	23.72599	20.24730	23.71899	400	0	400	0	0.216833
3	117.79886	145.63259	136.56855	2.33555	23.79187	23.44940	21.82843	400	0	0	400	0.076391
4	108.80012	134.50761	156.69227	2.12825	23.47333	22.78951	22.83461	400	0	400	0	0.076176
5	100.51214	154.73182	144.75604	1.96980	23.51711	23.72149	22.23780	400	0	0	400	0.054274
6	95.05696	146.33394	158.60909	1.84647	23.30981	23.23993	22.93045	400	0	0	400	0.008591
7	94.24034	145.07680	160.68286	1.82818	23.27895	23.16793	23.03414	400	0	0	400	0.001424
8	94.10611	144.87016	161.02373	1.82518	23.27388	23.15610	23.05119	400	0	0	400	0.000238
9	94.08374	144.83573	161.08052	1.82468	23.27304	23.15413	23.05403	400	0	0	400	0.000040
10	94.07999	144.82996	161.09005	1.82459	23.27289	23.15380	23.05450	400	0	0	400	0.000007
11	94.07936	144.82899	161.09165	1.82458	23.27287	23.15374	23.05458	400	0	0	400	0.000001
12	94.07925	144.82882	161.09192	1.82458	23.27287	23.15373	23.05460	400	0	0	400	

■ The Convergence Rate

Iteration	x1-hat	x1	x2
1	24.79328	24.79328	74.379837
2	21.68329	93.64305	21.68329
3	7.639072	7.639072	14.7352506
4	7.617623	15.03574	7.61762309
5	5.427376	5.427376	9.56992838
6	0.859089	0.859089	1.30746789
7	0.142436	0.142436	0.2121411
8	0.023765	0.023765	0.03526979
9	0.003989	0.003989	0.00591671
10	0.00067	0.00067	0.00099334
11	0.000112	0.000112	0.00016679



Conclusions

- The mathematical program to solve the equilibrium flow can only be formulated in a restrictive way
  - The modification of formulation must be careful in order to keep the equivalency conditions held
  - Adding a constraint concerning the flow variable would ruin the equivalency conditions

- The proposed algorithm can be seen as an approximate method
  - A more convincing approach is to apply bi-level programming method to analyze the interaction
- When a large network is applied, the computational works could be extensive
  - The streamlined approach employed in the algorithm would be advantageous for saving the effort of all-or-nothing assignment and shortest path search

Thank You