MAPL 699 Optimization and Equilibrium Problems
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An Overview of the Nonlinear Complementarity and Variational Inequality Problems

Background

Within the last few years, the study of optimization problems has been extended to include problems which find an equilibrium in a system under consideration. The notion of equilibrium is very general and includes for example, the Karush-Kuhn-Tucker (KKT) optimality conditions for nonlinear programs. Other examples in engineering and economics include problems involving nonlinear obstacles, contact mechanics, structural mechanics, elastohydrodyanamic lubrication, traffic equilibrium, Walrasian economic equilibrium, game theory, and energy economic equilibrium. These problems are usually analyzed using the variational inequality problem (VIP) or nonlinear complementarity problem (NCP) format as described below. For an extensive discussion of application areas see the survey article by Ferris and Pang (1997).

The Mathematical Statement of the VIP and the NCP

Although the engineering and economic problems mentioned above arise from rather diverse settings, they share a common mathematical format. These equilibrium problems are instances of the nonlinear complementarity problem (NCP) or the variational inequality problem (VIP) defined as follows. Having a function $F: R^n \to R^n$, the (pure) nonlinear complementarity problem NCP(F), is to find a vector $x^* \in R^n$ such that:

$$F(x^*) \geq 0,\ x^* \geq 0,\ F(x^*)^T x^* = 0,$$

where the inequalities for F and x* are componentwise.

There is also another version of the nonlinear complementarity problem, called the mixed NCP and denoted as MNCP(F) which can be stated as follows: find a vector $x^* \in R^n$ such that:

$$\begin{cases} x_i^* = u_i \quad \Rightarrow F_i(x^*) \leq 0 \\ l_i < x_i^* < u_i \Rightarrow F_i(x^*) = 0 \\ x_i^* = l_i \quad \Rightarrow F_i(x^*) \geq 0 \end{cases}$$

where

$l, u \in R^n$

are lower and upper bound vectors, respectively, and

$l_i, u_i \in R \cup \{-\infty, +\infty\}, l_i < u_i, \forall i = 1,2,\ldots,n$

It is not hard to see that for $u_i = +\infty, l_i = 0$ MNCP(F) reduces to the pure NCP. Thus, the pure NCP is a special case of the mixed one. In some cases the distinction between these forms of the problem is important.

For the variational inequality problem, in addition to the function F as defined above, there is a nonempty set $X \subseteq R^n$. VIP(F,X) is to find a vector $x^* \in X$ such that:

$$F(x^*)^T (y - x^*) \geq 0, \ \forall y \in X .$$

Both the nonlinear complementarity problem and the variational inequality problem are related to each other and for the purposes of MAPL 699 we will discuss them together as "equilibrium problems"; see Harker and Pang (1990) for more details on the relation between the NCP and the VIP problems.

REFERENCES

M. C. Ferris and J. S. Pang, "Engineering and economic applications of complementarity problems," *SIAM Review*, vol. 39, No.4, pp. 669-713, December 1997.

P. T. Harker and J. S. Pang, "Finite-dimensional variational inequality and nonlinear complementarity problems: a survey of theory, algorithms,and applications," *Mathematical Programming*, 48, pp. 161-220, 1990.